Inverse Trigonometric Functions

Multiple Choice Questions

Q: 1 **Which of the following is equal to $-\tan^{-1}\left(\frac{2}{3\pi}\right)$?**

1 $\cot^{-1}\left(\frac{3\pi}{2}\right)$

2 $-\cot^{-1}\left(\frac{3\pi}{2}\right)$

3 $\frac{\pi}{2} - \cot^{-1}\left(\frac{3\pi}{2}\right)$

4 $\frac{\pi}{2} + \cot^{-1}\left(\frac{3\pi}{2}\right)$

Free Response Questions

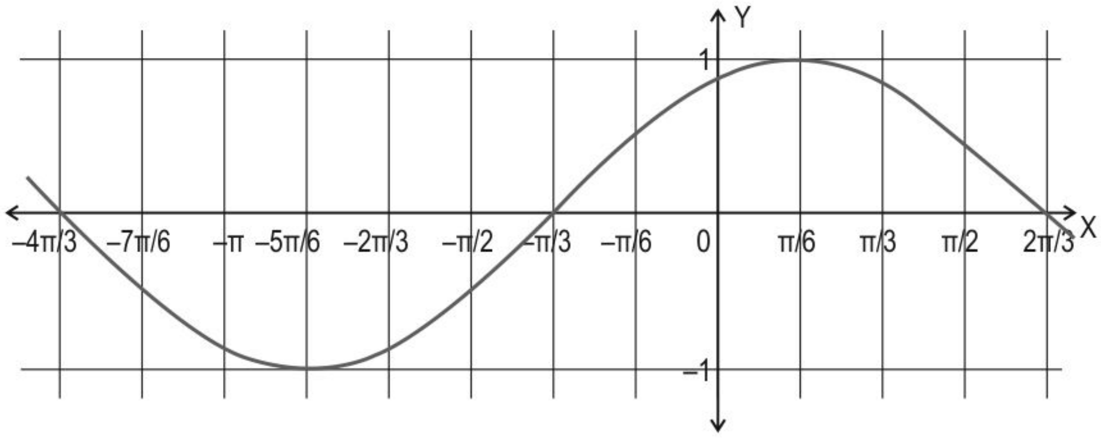
Q: 2 **Draw the graph of $\cos^{-1}(2x)$ in the domain $[-\frac{1}{2}, \frac{1}{2}]$.** **[1]**

Q: 3 **State whether the following statements are true or false. Explain your reasoning.** **[2]**

i) $\cos(\sec(x))^{-1} = \frac{1}{x}$

ii) $\sin^{-1}(-x) + \cos^{-1}(-x) = \sin^{-1}(x) - \cos^{-1}(x)$

Q: 4 **Shown below is a portion of the graph for $\sin(x + a)$, where $a \in R$ and $a > 0$.** **[2]**



What should the domain of the function be restricted to such that the function is invertible? Give valid reasons for your answer.

Q: 5 **If $4x + \frac{1}{x} = 4$, $x \in [0, \frac{\pi}{2}]$, find $\cos^{-1}x - \sin^{-1}x$. Show your work.** **[2]**

Q: 6 Show that $\cot^{-1}\left(\frac{1-x^2}{2x}\right) + \tan^{-1}\left(\frac{x^3-3x}{1-3x^2}\right) = -\tan^{-1}x.$ **[3]**

Q: 7 $f(x) = \cos(2\sin^{-1} x)$ **and** $g(x) = \sin^2(2\cos^{-1} x)$**,** $-1 \leq x \leq 1$**.** **[3]**

If $h(x) = (f(x) + g(x))$**, find** $h(0.1)$**. Show your work.**

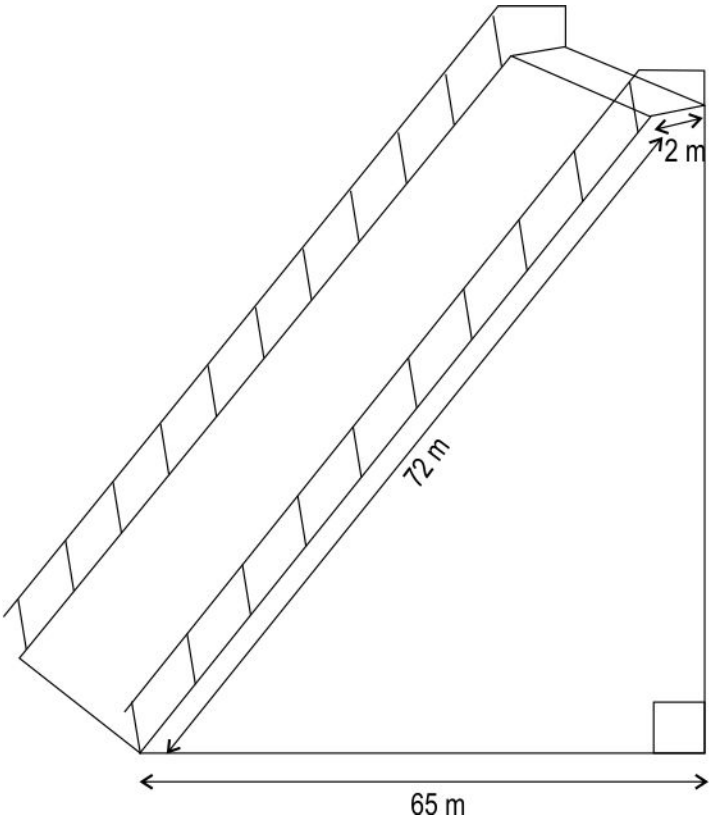
Q: 8 $\text{cosec}\,\theta = \frac{1}{\sin\theta}$ **[3]**

For $y \in \mathrm{R}$ where $\text{cosec}^{-1} y$ and $\sin^{-1} y$ are defined,
is $\text{cosec}^{-1} y = \frac{1}{\sin^{-1} y}$? If not, write the correct statement.

Q: 9 **Find the domain and range of the function** $y = \sin^{-1}(|x| - 1)$**. Show your work and give valid reasons.** **[3]**

Q: 10 **As per guidelines issued by the Home Ministry of India, in order for a ramp to be accessible to persons with disability, it must have a minimum slope ratio of 1:12. That is, for every unit of height, there must be at least 12 units of ramp.** **[3]**
(Source: https://www.mha.gov.in/sites/default/files/PublicNoticeforDraftStandards_22112021.pdf)

Abbas measured the dimensions of a ramp at his school and noted them in a rough diagram, as shown in the figure below.



(Note: The figure is not to scale.)

Is the given ramp meeting the Home Ministry's accessibility standards? Justify your answer.

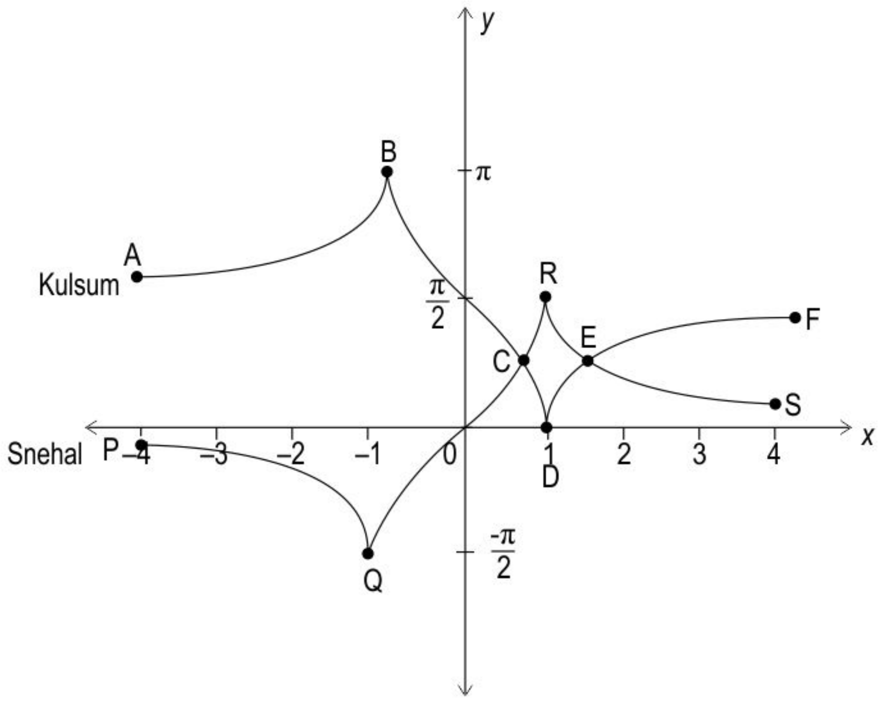
(Note: Assume that the top of the ramp is parallel to the ground.)

Q: 11 Evaluate $\tan\left(\frac{1}{2}\cos^{-1}\frac{5}{7}\right)$. Show your work. **[5]**

Case Study

Answer the questions based on the given information.
Madhu created a design on his floor using a combination of graphs of inverse trigonometric functions in the domain [-4, 4]. He also represented the coordinate axes for reference. He asked his friends, Kulsum and Snehal, to choose a path to walk on. Kulsum chose path ABCDEF, while

Snehal chose path PQCRES. They both started walking at the same time and with the same speed.



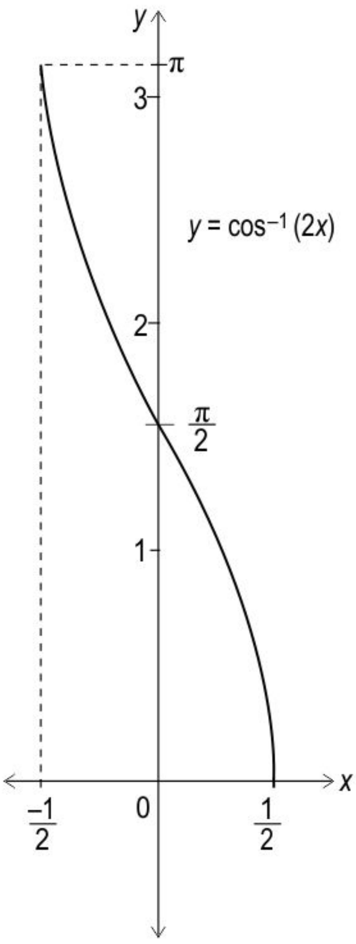
Q: 12 **Write the range of each of the two functions that Kulsum chose as her path to walk on.** **[2]**

Q: 13 **The graphs of which of the two functions are combined to form the path that Snehal chose?** **[1]**

Q: 14 **What is the *x* -coordinate of Kulsum's and Snehal's first meeting point? Show your steps.** **[2]**

The table below gives the correct answer for each multiple-choice question in this test.

Q.No	Correct Answers
1	2

Q.No	What to look for	Marks
2	Draws the graph of $\cos^{-1}(2x)$ in the given domain as follows: 	1
3	i) Writes that the given statement is false.	0.5
	Gives reason that $(\sec(x))^{-1} = \frac{1}{\sec x}$ and not $\sec^{-1}(x)$.	0.5
	ii) Writes that the given statement is false.	0.5

Q.No	What to look for	Marks
	Gives reason that $\sin^{-1}(-x) = -\sin^{-1}(x)$, and $\cos^{-1}(-x) = \cos^{-1}(x)$. **So, $\sin^{-1}(-x) + \cos^{-1}(-x) = \cos^{-1}(x) - \sin^{-1}(x)$.**	**0.5**
4	**Writes that the domain should be restricted to $[\frac{-5\pi}{6}, \frac{\pi}{6}]$.** **(Award full marks for any domain that satisfies the conditions.)**	**1**
	Reasons that if its domain is restricted to $[\frac{-5\pi}{6}, \frac{\pi}{6}]$, then the function becomes one-one and onto and hence, invertible.	**1**
5	**Rewrites $4x + \frac{1}{x} = 4$ as $(2x - 1)^2 = 0$.** **Hence, finds the value of x as $\frac{1}{2}$.**	**1**
	Evaluates $\cos^{-1} x - \sin^{-1} x$ as $\frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$ or 30°.	**1**
6	**Writes that:** $\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$	**0.5**
	Takes $x = \tan\theta$. Then $\theta = \tan^{-1} x$. **Solves the LHS by substituting the expression from step 1 as follows:** $\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \tan^{-1}\left(\frac{\tan^3\theta - 3\tan\theta}{1-3\tan^2\theta}\right)$ $= \tan^{-1}(\tan 2\theta) + \tan^{-1}(-\tan 3\theta)$	**1.5**
	Finds $\tan^{-1}(-\tan(3\theta)) = \tan^{-1}(\tan(-3\theta))$.	**0.5**
	Simplifies the expression from step 2 as $\tan^{-1}(\tan(2\theta)) + \tan^{-1}(\tan(-3\theta))$ **$= 2\theta - 3\theta = -\theta = -\tan^{-1} x$.**	**0.5**

Q.No	What to look for	Marks
7	**Using the formula cos 2θ = 1 - 2sin² 2θ, writes $f(x)$ as follows:** **$\cos(2\sin^{-1} x) = 1 - 2\sin^2(\sin^{-1} x) = 1 - 2x^2$**	1
	Using the formula sin 2θ = 2sin θ.cos θ, writes $g(x)$ as follows: **$\sin^2(2\cos^{-1} x) = 4\sin^2(\cos^{-1} x).\cos^2(\cos^{-1} x)$** **$= 4\cos^2(\cos^{-1} x)(1 - \cos^2(\cos^{-1} x))$** **$= 4x^2(1 - x^2) = 4x^2 - 4x^4$**	1
	Finds $h(x) = (1 - 2x^2) + (4x^2 - 4x^4) = 1 + 2x^2 - 4x^4$	0.5
	Calculates $h(0.1)$ as $1 + 2(0.1)^2 - 4(0.1)^4 = 1.0204$	0.5
8	Writes: $\text{cosec}^{-1} y = x$ $\Rightarrow \text{cosec}\, x = y$ $\Rightarrow \frac{1}{\sin x} = y$ $\Rightarrow \sin x = \frac{1}{y}$ $\Rightarrow \sin^{-1}\frac{1}{y} = x = \text{cosec}^{-1} y$	1.5
	Uses the given relation and the above step to write: $\sin^{-1}\frac{1}{y} = \frac{1}{\sin^{-1} y}$ which is not true for $y = 1$. Hence, $\text{cosec}^{-1} y = \frac{1}{\sin^{-1} y}$ is not true.	1.5
9	**Since the domain of inverse of sine function is [-1, 1], finds the domain of the given function as follows:** **$-1 \leq \lvert x \rvert - 1 \leq 1$** **$\Rightarrow 0 \leq \lvert x \rvert \leq 2$** **$\Rightarrow \lvert x \rvert \leq 2$** **$\Rightarrow -2 \leq x \leq 2$**	1.5

Q.No	What to look for	Marks
	Concludes the domain of $\sin^{-1}(\lvert x \rvert - 1)$ as [-2, 2].	**0.5**
	Finds the range of the function as an interval of real numbers where sin *y* is bijective. **Some examples are:** $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $[-\frac{3\pi}{2}, -\frac{\pi}{2}]$, ...	**0.5**
	Gives a valid reason. For example, since sin *y* is bijective in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the range of the given function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.	**0.5**
10	**Assumes the angle of inclination of the given ramp as θ.** **Finds the base length of the given ramp as 63 m, and hence finds θ as $\cos^{-1}\frac{63}{72} = \cos^{-1}\frac{7}{8}$.**	**0.5**
	Notes that, for a ramp to meet the given accessibility standards, the angle of inclination must be less than or equal to $\sin^{-1}\frac{1}{12}$.	**0.5**
	Writes $\sin^{-1}\frac{1}{12}$ as follows: $\sin^{-1}\left(\frac{1}{12}\right) = \cos^{-1}\left(\sqrt{1-\left(\frac{1}{12}\right)^2}\right)$ $\Rightarrow \sin^{-1}\left(\frac{1}{12}\right) = \cos^{-1}\left(\frac{\sqrt{143}}{12}\right)$	**1**
	Notes that : $\cos^{-1}\left(\frac{7}{8}\right) > \cos^{-1}\left(\frac{\sqrt{143}}{12}\right)$ **Gives a valid reason. For example, the value of cos θ decreases as θ increases, when 0° ≤ θ ≤ 90°.** **Concludes that the ramp is not meeting the accessibility standards set by the Home Ministry.** **(Award no marks if student simply writes that the ramp does not meet the accessibility standards, if no valid reason is given.)**	**1**

Q.No	What to look for	Marks
11	Assumes that $\frac{1}{2}\cos^{-1}\frac{5}{7} = \theta$. Then, $\cos^{-1}\frac{5}{7} = 2\theta$ $\Rightarrow \cos 2\theta = \frac{5}{7}$	**0.5**
	Writes cos 2θ as follows: $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{5}{7}$	**1**
	Solves the equation from step 3 to get: $\tan^2\theta = \frac{1}{6}$ $\Rightarrow \tan\theta = \left(\sqrt{\frac{1}{6}}\right)$ or $\left(-\sqrt{\frac{1}{6}}\right)$	**1**
	Writes that $0 \leq \cos^{-1}\frac{5}{7} \leq \pi$. $\Rightarrow 0 \leq \frac{1}{2}\cos^{-1}\frac{5}{7} \leq \frac{\pi}{2}$ $\Rightarrow 0 \leq \tan\theta \leq \infty$	**2**
	Rejects the negative value for tan θ, and writes that: $\tan\theta = \frac{1}{\sqrt{6}}$	**0.5**
12	**Identifies two functions that Kulsum chose to walk on as cos $^{-1}$ (*x*) and sec $^{-1}$ (*x*).**	**1**
	Writes that the range of cos $^{-1}$ (*x*) is [0, π].	**0.5**
	Writes that the range of sec $^{-1}$ (*x*) is [0, π] - $\frac{\pi}{2}$.	**0.5**
13	**Identifies the trigonometric functions that Snehal chose to walk on as sin $^{-1}$ (*x*) and cosec $^{-1}$ (*x*).**	**1**

Q.No	What to look for	Marks
14	Mentions that both friends met at point C which is the point of intersection of two functions, $\sin^{-1}(x)$ and $\cos^{-1}(x)$ and writes: $\cos^{-1}(x) = \sin^{-1}(x)$	0.5
	Simplifies the above equation as: $\cos(\cos^{-1}(x)) = \cos(\sin^{-1}(x))$ $\Rightarrow x = \cos(\sin^{-1}(x))$ Substitutes $\sin^{-1}(x)$ as u and simplifies the above equation as: $\sin u = \cos u$ $\Rightarrow \sin u = \sin(\frac{\pi}{2} - u)$ $\Rightarrow u = \frac{\pi}{4}$ (Award full marks if $\cos^{-1}(x)$ and $\sin^{-1}(x)$ are shown to be equal at $y = \frac{\pi}{4}$ directly.)	1
	Finds the x -coordinate of Kulsum's and Snehal's first meeting point as: $x = \sin u = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$	0.5